

# On a subclass of certain $k$ -starlike functions with negative coefficients

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**Abstract.** The aim of the present paper is to show some properties of functions belonging to the class  $(k, n, \alpha) - \mathcal{ST}$ . The obtained results extend the results by Silverman [3].

**Keywords:** univalent, starlike,  $k$ -starlike.

**Mathematical subject classification:** 30C45, 30C50.

## 1 Introduction

Denote by  $\mathcal{H}$  a class of functions of the form

$$f(z) = z + a_2 z^2 + \dots \quad (1.1)$$

analytic in the open unit disk  $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ , by  $S$  the class of functions (1.1), analytic and univalent in  $\mathbb{U}$ , by  $\mathcal{ST}(\alpha)$  subclass consisting of starlike and univalent functions of order  $\alpha$  and by  $k - \mathcal{ST}$  ( $0 \leq k < \infty$ ) a class of  $k$ -starlike univalent functions in  $\mathbb{U}$ , introduced in [2] and investigated Lecko and Wisniowska in [1].

It is known that every  $f \in k - \mathcal{ST}$  has a continuous extension to  $\overline{\mathbb{U}}$ ,  $f(\mathbb{U})$  is bounded and  $f(\partial\mathbb{U})$  is a rectifiable curve [2].

**Lemma 1.1. [1]** Let  $f \in S$  and  $0 \leq k < \infty$ . Then  $f \in k - \mathcal{ST}$  iff

$$\operatorname{Re} \left\{ \frac{\zeta}{z} + \frac{(z - \zeta) f'(z)}{f(z)} \right\} \geq 0 \quad (1.2)$$

$$z \in \mathbb{U}, |\zeta| \leq k.$$

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Let  $\mathcal{T}$  denote the subclass consisting of functions whose nonzero coefficients, from the second on, are negative. That is, an analytic and univalent function  $f$  is in  $\mathcal{T}$  if it can be expressed as

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad n \in \mathbb{N}. \quad (1.3)$$

In [3], Silverman introduced the subclass of  $\mathcal{T}$  denoted by  $\mathcal{T}^*(\alpha)$  which consists of functions, that are starlike of order  $\alpha$ .

Denote by  $\mathcal{A}(n)$  the class of functions of the form

$$f(z) = z - \sum_{m=n+1}^{\infty} a_m z^m, \quad a_m \geq 0, \quad n \in \mathbb{N} \quad (1.4)$$

that are analytic in the open unit disc  $\mathbb{U}$ . In the present paper, a subclass  $(k, n, \alpha) - S\mathcal{T}$  of starlike functions in the open unit disc  $\mathbb{U}$  is introduced. A function  $f(z) \in \mathcal{A}(n)$  is said to be in the class  $(k, n, \alpha) - S\mathcal{T}$  if it satisfies

$$\operatorname{Re} \left\{ \frac{\zeta}{z} + \frac{(z - \zeta) f'(z)}{f(z)} \right\} \geq \alpha \quad (1.5)$$

for some  $\alpha$  ( $0 \leq \alpha < 1$ ),  $z \in \mathbb{U}$ , and  $|\zeta| \leq k$ ,  $k \geq 0$ .

We note that  $(0, 1, \alpha) - S\mathcal{T} \equiv \mathcal{T}^*(\alpha)$  and  $(k, 1, 0) - S\mathcal{T} \equiv k - S\mathcal{T} \cap \mathcal{A}(n)$ . Our class  $(k, n, \alpha) - S\mathcal{T}$  is the generalization of the class  $\mathcal{T}^*(\alpha)$  introduced by Silverman in [3].

## 2 Some results of the class $(k, n, \alpha) - S\mathcal{T}$

**Theorem 2.1.** *A function  $f \in \mathcal{A}(n)$  is in the class  $(k, n, \alpha) - S\mathcal{T}$  iff*

$$\sum_{m=n+1}^{\infty} [k(m-1) + m - \alpha] a_m \leq 1 - \alpha. \quad (2.1)$$

**Proof.** Let  $f \in (k, n, \alpha) - S\mathcal{T}$ . Then we have from (1.5)

$$\begin{aligned} & \operatorname{Re} \left\{ \frac{\zeta}{z} + \frac{(z - \zeta) f'(z)}{f(z)} \right\} \\ &= \operatorname{Re} \left\{ \frac{z - \sum_{m=n+1}^{\infty} m a_m z^m - \zeta \sum_{m=n+1}^{\infty} a_m z^{m-1} + \zeta \sum_{m=n+1}^{\infty} m a_m z^{m-1}}{z - \sum_{m=n+1}^{\infty} a_m z^m} \right\} \geq \alpha. \end{aligned}$$

If we choose  $z$  and  $\zeta$  real and let  $z \rightarrow 1^-$  and  $\zeta \rightarrow -k^+$ , we get,

$$\frac{1 - \sum_{m=n+1}^{\infty} m a_m + k \sum_{m=n+1}^{\infty} a_m - k \sum_{m=n+1}^{\infty} m a_m}{1 - \sum_{m=n+1}^{\infty} a_m} \geq \alpha$$

or

$$\sum_{m=n+1}^{\infty} [k(m-1) + m - \alpha] a_m \leq 1 - \alpha$$

which is equivalent to (2.1).

Conversely, assume that (2.1) is true. Then

$$\begin{aligned} & \left\{ \frac{\zeta}{z} + \frac{(z - \zeta) f'(z)}{f(z)} \right\} \\ &= \left\{ \frac{\zeta}{z} + \frac{(z - \zeta) \left( 1 - \sum_{m=n+1}^{\infty} m a_m z^{m-1} \right)}{z - \sum_{m=n+1}^{\infty} a_m z^m} \right\} \\ &= \left\{ \frac{1 - \sum_{m=n+1}^{\infty} m a_m z^{m-1} + \zeta \sum_{m=n+1}^{\infty} (m-1) a_m z^{m-2}}{1 - \sum_{m=n+1}^{\infty} a_m z^{m-1}} \right\} \end{aligned}$$

for  $|z| < 1$ . If we choose  $z \rightarrow 1^-$  and  $\zeta \rightarrow -k^+$  through real values, we obtain

$$\operatorname{Re} \left\{ \frac{\zeta}{z} + \frac{(z - \zeta) f'(z)}{f(z)} \right\} = \frac{1 - \sum_{m=n+1}^{\infty} [k(m-1) + m] a_m}{1 - \sum_{m=n+1}^{\infty} a_m}. \quad (2.2)$$

If (2.1) is rewritten as

$$\sum_{m=n+1}^{\infty} [km + m - k] a_m \leq 1 - \alpha + \alpha \sum_{m=n+1}^{\infty} a_m,$$

and (2.2) is used, then we obtain

$$\operatorname{Re} \left\{ \frac{\zeta}{z} + \frac{(z - \zeta) f'(z)}{f(z)} \right\} \geq \frac{\alpha (1 - \sum_{m=n+1}^{\infty} a_m)}{1 - \sum_{m=n+1}^{\infty} a_m} = \alpha.$$

Thus  $f \in (k, n, \alpha) - \mathcal{ST}$ .

**Theorem 2.2.** *If  $f \in (k, n, \alpha) - \mathcal{ST}$ , then we obtain*

$$r - \frac{1 - \alpha}{(1 + k)n + (1 - \alpha)} r^{n+1} \leq |f(z)| \leq r + \frac{1 - \alpha}{(1 + k)n + (1 - \alpha)} r^{n+1}$$

for  $|z| = r$ , with equality for

$$f(z) = z - \frac{1 - \alpha}{(1 + k)n + (1 - \alpha)} z^{n+1}; \quad z = \mp r$$

**Proof.** From (2.1), we have

$$\sum_{m=n+1}^{\infty} a_m \leq \frac{1 - \alpha}{(1 + k)n + (1 - \alpha)}. \quad (2.3)$$

Thus,

$$|f(z)| \leq r + \sum_{m=n+1}^{\infty} a_m r^m \leq r + r^{n+1} \sum_{m=n+1}^{\infty} a_m \leq r + \frac{1 - \alpha}{(1 + k)n + (1 - \alpha)} r^{n+1}.$$

Similarly,

$$|f(z)| \geq r - \sum_{m=n+1}^{\infty} a_m r^m \geq r - r^{n+1} \sum_{m=n+1}^{\infty} a_m \geq r - \frac{1 - \alpha}{(1 + k)n + (1 - \alpha)} r^{n+1}.$$

**Theorem 2.3.** If  $f \in (k, n, \alpha) - \mathcal{ST}$ , then

$$1 - \frac{(1 - \alpha)(n + 1)}{(1 + k)n + (1 - \alpha)} r^n \leq |f'(z)| \leq 1 + \frac{(1 - \alpha)(n + 1)}{(1 + k)n + (1 - \alpha)} r^n$$

for  $|z| = r$ , with equality for

$$f(z) = z - \frac{(1 - \alpha)(n + 1)}{(1 + k)n + (1 - \alpha)} z^{n+1} \quad ; \quad z = \mp r$$

**Proof.** From (2.3) and Theorem 2.1, it follows that

$$\sum_{m=n+1}^{\infty} m a_m \leq \frac{(1 - \alpha)(n + 1)}{(1 + k)n + (1 - \alpha)}.$$

Consequently, for  $|z| = r < 1$ , we have

$$\begin{aligned} |f'(z)| &\leq 1 + \sum_{m=n+1}^{\infty} m a_m |z|^{m-1} \leq 1 + r^n \sum_{m=n+1}^{\infty} m a_m \\ &\leq 1 + \frac{(1 - \alpha)(n + 1)}{(1 + k)n + (1 - \alpha)} r^n \end{aligned}$$

and

$$\begin{aligned} |f'(z)| &\geq 1 - \sum_{m=n+1}^{\infty} m a_m |z|^{m-1} \geq 1 - r^n \sum_{m=n+1}^{\infty} m a_m \\ &\geq 1 - \frac{(1-\alpha)(n+1)}{(1+k)n + (1-\alpha)} r^n. \end{aligned}$$

This completes the proof of the theorem.

**Theorem 2.4.** *Let the functions*

$$f(z) = z - \sum_{m=n+1}^{\infty} a_m z^m, \quad a_m \geq 0$$

and

$$g(z) = z - \sum_{m=n+1}^{\infty} b_m z^m, \quad b_m \geq 0$$

be in the class  $(k, n, \alpha) - \mathcal{ST}$ . Then for  $0 \leq \lambda \leq 1$ ,

$$h(z) = (1-\lambda)f(z) + \lambda g(z) = z - \sum_{m=n+1}^{\infty} c_m z^m, \quad c_m \geq 0$$

is in the class  $(k, n, \alpha) - \mathcal{ST}$ .

**Proof.** Assume that  $f, g \in (k, n, \alpha) - \mathcal{ST}$ . Then we have from Theorem 2.1

$$\sum_{m=n+1}^{\infty} [k(m-1) + m - \alpha] a_m \leq 1 - \alpha$$

and

$$\sum_{m=n+1}^{\infty} [k(m-1) + m - \alpha] b_m \leq 1 - \alpha.$$

Therefore, we can see that

$$\begin{aligned} &\sum_{m=n+1}^{\infty} [k(m-1) + m - \alpha] c_m \\ &= \sum_{m=n+1}^{\infty} [k(m-1) + m - \alpha] [(1-\lambda)a_m + \lambda b_m] \end{aligned}$$

$$\begin{aligned}
&= (1 - \lambda) \sum_{m=n+1}^{\infty} [k(m-1) + m - \alpha] a_m \\
&\quad + \lambda \sum_{m=n+1}^{\infty} [k(m-1) + m - \alpha] b_m \\
&\leq (1 - \lambda)(1 - \alpha) + \lambda(1 - \alpha) = (1 - \alpha)
\end{aligned}$$

which completes the proof of Theorem 2.4.

**Definition 2.1.** The modified Hadamard product  $f * g$  of two functions

$$f(z) = z - \sum_{m=n+1}^{\infty} a_m z^m, \quad (a_m \geq 0) \quad \text{and} \quad g(z) = z - \sum_{m=n+1}^{\infty} b_m z^m, \quad (b_m \geq 0)$$

is denoted by

$$(f * g)(z) = z - \sum_{m=n+1}^{\infty} a_m b_m z^m.$$

We now prove the following.

**Theorem 2.5.** If  $f, g \in (k, n, \alpha) - ST$ , then  $(f * g) \in (k, n, \beta) - ST$ , where

$$\beta = \frac{(1+k)n + 2(1-\alpha) - (1-\alpha)^2}{(1+k)n + 2(1-\alpha)}.$$

The result is sharp for the functions  $f(z)$  and  $g(z)$  given by

$$f(z) = g(z) = z - \frac{(1-\alpha)}{(1+k)n + (1-\alpha)} z^{n+1}$$

where  $0 \leq \alpha < 1$  and  $0 \leq k < \infty$ .

**Proof.** From Theorem 2.1, we have

$$\begin{aligned}
&\sum_{m=n+1}^{\infty} \frac{[k(m-1) + m - \alpha]}{1 - \alpha} a_m \leq 1 \quad \text{and} \\
&\sum_{m=n+1}^{\infty} \frac{[k(m-1) + m - \alpha]}{1 - \alpha} b_m \leq 1.
\end{aligned} \tag{2.4}$$

We have to find the largest  $\beta$  such that

$$\sum_{m=n+1}^{\infty} \frac{[k(m-1) + m - \beta]}{1 - \beta} a_m b_m \leq 1. \tag{2.5}$$

From (2.4), we find that

$$\sum_{m=n+1}^{\infty} \frac{[k(m-1) + m - \alpha]}{1 - \alpha} \sqrt{a_m b_m} \leq 1. \quad (2.6)$$

Therefore (2.5) is true if

$$\frac{[k(m-1) + m - \beta]}{1 - \beta} a_m b_m \leq \frac{[k(m-1) + m - \alpha]}{1 - \alpha} \sqrt{a_m b_m}$$

or

$$\sqrt{a_m b_m} \leq \frac{1 - \beta}{1 - \alpha} \frac{[k(m-1) + m - \alpha]}{[k(m-1) + m - \beta]}.$$

Note that from (2.6)

$$\sqrt{a_m b_m} \leq \frac{1 - \alpha}{[k(m-1) + m - \alpha]}.$$

Thus if

$$\frac{1 - \alpha}{[k(m-1) + m - \alpha]} \leq \frac{1 - \beta}{1 - \alpha} \frac{[k(m-1) + m - \alpha]}{[k(m-1) + m - \beta]}$$

or, equivalently, if

$$\beta \leq \frac{[k(m-1) + m - \alpha]^2 - (1 - \alpha)^2 [k(m-1) + m]}{[k(m-1) + m - \alpha]^2 - (1 - \alpha)^2}$$

then (2.5) is satisfied. Defining the function  $\Theta(m)$  by

$$\Theta(m) = \frac{[k(m-1) + m - \alpha]^2 - (1 - \alpha)^2 [k(m-1) + m]}{[k(m-1) + m - \alpha]^2 - (1 - \alpha)^2}$$

we can see that  $\Theta(m)$  is an increasing function of  $m$ . Therefore,

$$\beta \leq \Theta(n+1) = \frac{(1+k)n + 2(1-\alpha) - (1-\alpha)^2}{(1+k)n + 2(1-\alpha)}$$

which completes the assertion of theorem.

### 3 Extreme points for $(k, n, \alpha) - \mathcal{ST}$

**Theorem 3.1.** Let  $f_n(z) = z$  and  $f_m(z) = z - \frac{1-\alpha}{k(m-1)+m-\alpha} z^m$ ,  $m = n+1, n+2, \dots$ . Then  $f \in (k, n, \alpha) - \mathcal{ST}$  iff it can be expressed in the form

$$f(z) = \sum_{m=n}^{\infty} \zeta_m f_m(z),$$

where  $\zeta_m \geq 0$  for  $m \geq n$  and  $\sum_{m=n}^{\infty} \zeta_m = 1$ .

**Proof.** Assume that

$$f(z) = \sum_{m=n}^{\infty} \zeta_m f_m(z).$$

Then

$$\begin{aligned} f(z) &= \zeta_n f_n(z) + \sum_{m=n+1}^{\infty} \zeta_m f_m(z) \\ &= \zeta_n z + \sum_{m=n+1}^{\infty} \zeta_m z - \sum_{m=n+1}^{\infty} \zeta_m \frac{1-\alpha}{k(m-1)+m-\alpha} z^m \\ &= \left( \sum_{m=n}^{\infty} \zeta_m \right) z - \sum_{m=n+1}^{\infty} \zeta_m \frac{1-\alpha}{k(m-1)+m-\alpha} z^m \\ &= z - \sum_{m=n+1}^{\infty} \zeta_m \frac{1-\alpha}{k(m-1)+m-\alpha} z^m. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{m=n+1}^{\infty} \zeta_m \left( \frac{1-\alpha}{k(m-1)+m-\alpha} \right) \left( \frac{k(m-1)+m-\alpha}{1-\alpha} \right) \\ &= \sum_{m=n+1}^{\infty} \zeta_m = \sum_{m=n}^{\infty} \zeta_m - \zeta_n = 1 - \zeta_n \leq 1. \end{aligned}$$

We have  $f \in (k, n, \alpha) - \mathcal{ST}$ .

Conversely, suppose that  $f \in (k, n, \alpha) - \mathcal{ST}$ . Since

$$|a_m| \leq \frac{1-\alpha}{k(m-1)+m-\alpha} \quad , \quad m = n+1, n+2, \dots,$$

we can set

$$\zeta_m = \frac{k(m-1)+m-\alpha}{1-\alpha} \quad , \quad m = n+1, n+2, \dots,$$

and

$$\zeta_n = 1 - \sum_{m=n+1}^{\infty} \zeta_m.$$

Then

$$\begin{aligned}
 f(z) &= z - \sum_{m=n+1}^{\infty} a_m z^m \\
 &= z - \sum_{m=n+1}^{\infty} \frac{1-\alpha}{k(m-1)+m-\alpha} \zeta_m z^m \\
 &= z - \sum_{m=n+1}^{\infty} \zeta_m (z - f_m(z)) \\
 &= \left(1 - \sum_{m=n+1}^{\infty} \zeta_m\right) z + \sum_{m=n+1}^{\infty} \zeta_m f_m(z) \\
 &= \zeta_n z + \sum_{m=n+1}^{\infty} \zeta_m f_m(z) \\
 &= \zeta_n f_n(z) + \sum_{m=n+1}^{\infty} \zeta_m f_m(z) = \sum_{m=n}^{\infty} \zeta_m f_m(z).
 \end{aligned}$$

This completes the assertion of theorem.

**Corollary 3.1.** *The extreme points of  $(k, n, \alpha) - \mathcal{ST}$  are given by*

$$f_n(z) = z \quad \text{and} \quad f_m(z) = z - \frac{1-\alpha}{k(m-1)+m-\alpha} z^m, \quad m = n+1, n+2, \dots$$

If we take  $n = 1, k = 0$  in Corollary 3.1, then we have the following result by Silverman [3].

**Corollary 3.2.** *The extreme points of  $\mathcal{T}^*(\alpha)$  are given by*

$$f_1(z) = z \quad \text{and} \quad f_m(z) = z - \frac{(1-\alpha)}{m-\alpha} z^m, \quad m = 2, 3, \dots$$

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